

# Magnetic Field Generation from Self-Consistent Collective Neutrino-Plasma Interactions\*

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(February 1, 2008)

## Abstract

A new Lagrangian formalism for self-consistent collective neutrino-plasma interactions is presented in which each neutrino species is described as a classical ideal fluid. The neutrino-plasma fluid equations are derived from a covariant relativistic variational principle in which finite-temperature effects are retained. This new formalism is then used to investigate the generation of magnetic fields and the production of magnetic helicity as a result of collective neutrino-plasma interactions.

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\*This work was performed under Department of Energy Contracts No. PDDEFG-03-95ER-40936 and DE-AC03-76SF00098, in part by the National Science Foundation under grant PHY-95-14797, and in part also by Alfred P. Sloan Foundation.

## I. INTRODUCTION

Photons, neutrinos and plasmas are ubiquitous in the universe [1,2]. During the early universe, it is expected that photons and neutrinos interacted quite strongly with hot primordial plasmas [3]. Although photons and neutrinos decoupled from plasmas relatively early after the big bang [1,2], there are still conditions today where neutrino-plasma interactions might be important. For example, during a supernova explosion [4–6], intense neutrino fluxes are generated as result of the gravitational collapse of the stellar core. It is generally believed that the outgoing neutrino flux needs to transfer energy and momentum to the surrounding plasma in order to produce the observed explosion.

The self-consistent collective interaction between photons and plasmas is traditionally treated classically (i.e., without quantum-mechanical effects), where plasma particles are either treated within a fluid or a kinetic picture, while photons are described in terms of an electromagnetic field. For a self-consistent treatment of collective electromagnetic-plasma interactions (see Ref. [7], for example), one considers both the influence of electromagnetic fields on plasma dynamics and the generation of electromagnetic fields by plasma currents. The interaction between photons and neutrinos, on the other hand, requires a full quantum-mechanical treatment and has been the subject of recent interest [8].

Neutrino-plasma interactions involve charged and neutral currents associated with the weak force [9,10] (through the exchange of  $W^\pm$  and  $Z^0$  bosons, respectively). The collective interactions studied here apply to the intense neutrino fluxes. Discrete (i.e., collisional) neutrino-plasma interactions, on the other hand, involve scattering of individual particles; such discrete neutrino-plasma particle effects will be omitted in the present work.

The purpose of the present work is to investigate the self-consistent collective interaction between neutrinos and plasmas in the presence of electromagnetic fields. The inclusion of electromagnetic effects is a departure from conventional hydrodynamic models used in investigating neutrino interactions with astrophysical plasmas [5]. Here, we investigate the collective processes

$$EM \rightarrow \sigma \rightarrow \nu \quad (1.1)$$

and

$$\nu \rightarrow \sigma \rightarrow EM. \quad (1.2)$$

In the first process, the neutrino ( $\nu$ ) dynamics is influenced by an electromagnetic field ( $EM$ ) with a plasma ( $\sigma$ ) background acting as an intermediary, even though neutrinos are chargeless particles. In the second process, electromagnetic fields are generated as a result of plasma currents produced by neutrino ponderomotive effects. The problem of magnetic-field generation associated with self-consistent collective neutrino-plasma interactions is thus investigated here within the context of the process (1.2).

### A. Notation

In the present paper, the Latin subscript  $s$  refers to different components of the neutrino-plasma fluid: the subscript  $s = \nu$  refers to neutrinos while the subscript  $s = \sigma$  refers to

components of the plasma other than photons and neutrinos. To avoid confusion, we use the Greek letters  $\alpha, \beta, \dots$  for Lorentz indices rather than traditional  $\mu, \nu, \dots$ ; for example, the flux four-vector is  $J^\alpha = Nu^\alpha$ , with proper density  $N$  (Lorentz scalar) and normalized four-velocity  $u^\alpha = (u^0, \mathbf{u})$ . In certain cases, objects with Lorentz indices may not be covariant; for instance, the fluid velocity  $v^\alpha = u^\alpha/u^0$  is not covariant and the number density in a given frame  $n = Nu^0$  is not a Lorentz scalar. The symbols in bold face are three-vectors while those in Sans Serif are four-dimensional tensors (such as  $\mathbf{F}$  for the electromagnetic field strength  $F_{\alpha\beta}$ ). The dot  $\cdot$  describes the contraction of a Lorentz index or an inner product of two three-vectors if in bold face. Here, we employ the metric  $g_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$  and, hence,  $a \cdot b \equiv a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$ .

## B. Neutrino Descriptions for Collective Neutrino-Plasma Interactions

To study collective neutrino-plasma interactions, neutrinos can either be described in terms of Dirac spinor fields [9–13], Klein-Gordon scalar fields [14,15], classical non-relativistic fluids [16], or relativistic quasi-particles [17,18]. In all these descriptions, the interaction between neutrinos (of type  $\nu$ ) and plasma particles (of species  $\sigma$ ) is described in terms of an effective weak-interaction charge  $G_{\sigma\nu}$ . In general,  $G_{\sigma\nu}$  has the following property [11]:

$$G_{\sigma\nu} = -G_{\bar{\sigma}\nu} = -G_{\sigma\bar{\nu}} = G_{\bar{\sigma}\bar{\nu}}, \quad (1.3)$$

where  $\sigma$  ( $\bar{\sigma}$ ) denotes a matter (anti-matter) species and  $\nu$  ( $\bar{\nu}$ ) denotes a neutrino (anti-neutrino) species. The effective charge  $G_{\sigma\nu}$  depends on the Fermi weak-interaction constant  $G_F$  ( $\approx 9 \times 10^{-38}$  eV cm<sup>3</sup>), the Weinberg angle  $\theta_W$  ( $\sin^2 \theta_W \approx 0.23$  [10]), and the species  $\sigma$  and  $\nu$ . For example, for neutrinos interacting with unpolarized electrons ( $e$ ), protons ( $p$ ) and neutrons ( $n$ ), one finds [11]

$$G_{\sigma\nu} = \sqrt{2}G_F \left[ \delta_{\sigma e} \delta_{\nu e} + (I_\sigma - 2Q_\sigma \sin^2 \theta_W) \right], \quad (1.4)$$

where  $I_\sigma$  is the weak isotopic spin for particle species  $\sigma$  ( $I_e = I_n = -1/2$  and  $I_p = 1/2$ ) and  $Q_\sigma \equiv q_\sigma/e$  is the normalized electric charge. Here, the first term in (1.4) is due to charged weak currents (and thus applies only to electrons and electron-neutrinos), while the remaining terms are due to neutral weak currents (and thus apply to all species).

To assist us in investigating self-consistent collective neutrino-plasma interactions in the present work, all neutrino and particle species are treated as ideal classical fluids. For this purpose, we proceed with the classical fluid limit for plasma-particles in the Dirac description expressed in terms of the correspondence

$$\bar{\psi}_\sigma (\hat{\gamma}^\alpha / c) \psi_\sigma \rightarrow J_\sigma^\alpha \equiv (n_\sigma, \mathbf{J}_\sigma), \quad (1.5)$$

where  $\psi_\sigma$  is the Dirac spinor field for particle species  $\sigma$  (with  $\hat{\gamma}^\alpha$  denoting Dirac matrices) while  $n_\sigma$  and  $\mathbf{J}_\sigma \equiv n_\sigma \mathbf{v}_\sigma / c$  are the particle density and (normalized) particle flux for each plasma-fluid species  $\sigma$  in the lab reference frame, respectively. In this limit, the propagation of a neutrino test-particle of type  $\nu$  in a background plasma is determined by the effective potential [19]

$$V_\nu(\mathbf{x}, \mathbf{v}, t) \equiv \sum_\sigma G_{\sigma\nu} \left[ n_\sigma(\mathbf{x}, t) - \mathbf{J}_\sigma(\mathbf{x}, t) \cdot \frac{\mathbf{v}}{c} \right], \quad (1.6)$$

where  $(\mathbf{x}, \mathbf{v})$  denote the neutrino's position and velocity. We note that neutrino propagation in matter is a topic at the heart of the problem of neutrino oscillations in matter [20–22] and the solar neutrino problem [23]. Although the term  $\mathbf{J}_\sigma \cdot \mathbf{v}/c$  is a relativistic correction to  $n_\sigma$  in (1.6), we keep it for the following reason. For a primordial plasma with a single family of particles ( $s = \sigma$ ) and anti-particles ( $s = \bar{\sigma}$ ), we find from (1.3)

$$\left. \begin{aligned} \sum_{s=\sigma, \bar{\sigma}} G_{s\nu} n_s &= 0 \\ \sum_{s=\sigma, \bar{\sigma}} G_{s\nu} \mathbf{J}_s &= G_{\sigma\nu} (\mathbf{J}_\sigma - \mathbf{J}_{\bar{\sigma}}) \end{aligned} \right\}, \quad (1.7)$$

and thus the effective neutrino potential (1.6) becomes  $V_\nu = G_{\sigma\nu} (\mathbf{J}_{\bar{\sigma}} - \mathbf{J}_\sigma) \cdot \mathbf{v}/c$ , for each  $(\sigma, \bar{\sigma})$ -family. Hence, keeping this relativistic correction is necessary for the description of collective neutrino interactions with a primordial plasma [24]. The model presented here therefore retains all relativistic effects associated with the neutrino and plasma fluids.

For a self-consistent description of collective neutrino-plasma interactions in which neutrino ponderomotive effects on the background medium are included, we now use a similar classical-fluid correspondence for the neutrinos. The propagation of a plasma test-particle of species  $\sigma$  (with electric charge  $q_\sigma$ ) in a background medium composed of a neutrino fluid of type  $\nu$  and an electromagnetic field is determined by the potential

$$V_\sigma(\mathbf{x}, \mathbf{v}, t) \equiv \left[ q_\sigma \phi(\mathbf{x}, t) + \sum_\nu G_{\sigma\nu} n_\nu(\mathbf{x}, t) \right] - \left[ q_\sigma \mathbf{A}(\mathbf{x}, t) + \sum_\nu G_{\sigma\nu} \mathbf{J}_\nu(\mathbf{x}, t) \right] \cdot \frac{\mathbf{v}}{c}, \quad (1.8)$$

where  $n_\nu$  and  $\mathbf{J}_\nu \equiv n_\nu \mathbf{v}_\nu/c$  are the neutrino density and (normalized) neutrino flux in the lab reference frame, respectively,  $\phi$  and  $\mathbf{A}$  are the electromagnetic potentials, and  $(\mathbf{x}, \mathbf{v})$  denote the plasma-particle's position and velocity. It is interesting to note how the right side of (1.8) links the electrostatic scalar potential  $\phi$  and the neutrino density  $n_\nu$ , on the one hand, and the magnetic vector potential  $\mathbf{A}$  and the neutrino flux vector  $\mathbf{J}_\nu$ , on the other hand. We will henceforth refer to the approximation whereby  $\mathbf{J}_\sigma$  and  $\mathbf{J}_\nu$  are omitted in (1.6) and (1.8) as the *weak*-electrostatic (or non-relativistic) approximation.

Although we assume that each neutrino flavor has a finite mass, this assumption is not crucial to the development of our model; see Section II for a discussion of neutrino-fluid dynamics for arbitrary neutrino masses. Furthermore we shall ignore all quantum mechanical effects, including effects due to strong magnetic fields [25] (i.e., we assume  $B/B_{\text{QM}} \equiv \hbar\Omega_e/m_e c^2 \ll 1$ , where  $\Omega_e \equiv eB/m_e c$  is the electron gyrofrequency and  $B_{\text{QM}} \sim 4 \times 10^{13}$  G). Hence, although magnetic fields appear explicitly in our model, they are not considered strong enough to modify the form of the interaction potentials (1.6) and (1.8).

### C. Magnetic-Field Generation due to Neutrino-Plasma Interactions

An important application of the process (1.2) involves the prospect of generating magnetic fields in an unmagnetized plasma as a result of collective neutrino-plasma interactions.

This application may be of importance in investigating magnetogenesis in the early universe (e.g., see Ref. [2]). A similar process of magnetic-field generation occurs in laser-plasma interactions whereby an intense laser pulse propagating in a nonuniform plasma generates a quasi-static magnetic field. This process was first studied theoretically [26–28] and was recently confirmed experimentally [29].

The generation of magnetic fields by collective neutrino-plasma interactions was first contemplated in the non-relativistic (weak-electrostatic) limit by Shukla *et al.* [30,31]. The covariant (relativistic) Lagrangian approach introduced by Brizard and Wurtele [15], however, revealed the presence of additional ponderomotive terms missing from previous analysis [14,30,31]. These additional ponderomotive terms involve the time derivative of the neutrino flux  $\partial_t \mathbf{J}_\nu$  and the curl of the neutrino flux  $\nabla \times \mathbf{J}_\nu$  (henceforth referred to as the neutrino-flux vorticity), which are shown here to lead to significantly different predictions regarding neutrino-induced magnetic-field generation. In fact, we show that magnetic-field generation due to neutrino-plasma interactions is not possible without these new terms.

## D. Organization

The remainder of this paper is organized as follows. In Section II, the Lagrangian formalism for ideal fluids is introduced. In Section III, a variational principle for collective neutrino-plasma interactions in the presence of an electromagnetic field is presented. This Lagrangian formalism is fully relativistic and covariant and can thus be generalized to include general relativistic effects (e.g., see Refs. [32,33]). In Section IV, the nonlinear neutrino-plasma fluid equations and the Maxwell equations for the electromagnetic field are derived. Through the Noether method [34–36], an exact energy-momentum conservation law is also derived and the process of energy-momentum transfer from the neutrinos to the electromagnetic field and the plasma is discussed. In Section V, magnetic-field generation, magnetic-helicity production and magnetic equilibrium involving neutrino-plasma interactions are investigated. Here, we find that neutrino-flux vorticity ( $\nabla \times \mathbf{J}_\nu$ ) plays a fundamental role in all three processes. We summarize our work in Section VI and discuss future work.

## II. LAGRANGIAN DENSITY FOR A FREE IDEAL FLUID

The present Section is dedicated to the derivation of a suitable Lagrangian density for a free ideal fluid from an existing single-particle Lagrangian for a free particle of arbitrary mass (including zero). The difficulty with dealing with the case of free neutrinos as particles is that their mass may be zero. Since the relativistic Lagrangian  $L$  for a free single particle of mass  $m$  is [37]

$$L = -mc^2 \gamma^{-1} \equiv -mc \left( \frac{dx^\alpha}{dt} \frac{dx_\alpha}{dt} \right)^{1/2}, \quad (2.1)$$

it is not obvious how to handle the limiting case of zero mass. This difficulty is resolved in [38] as follows (see also Ref. [2]).

## A. Single-Particle Lagrangian

Consider the primitive Lagrangian

$$L_p = \mathbf{p} \cdot \dot{\mathbf{x}} - E \dot{t} \equiv -p_\alpha c v^\alpha \quad (2.2)$$

for a particle of arbitrary rest-mass  $m$  (including zero), where  $(x, p)$  are coordinates in the eight-dimensional phase space in which the particle moves and  $\dot{x}^\alpha = (c, \mathbf{v}) \equiv c v^\alpha$ . Although the particle's space-time location  $x^\alpha = (ct, \mathbf{x})$  is arbitrary, its four-momentum  $p^\alpha = (E/c, \mathbf{p})$  is not since the particle's physical motion is constrained to occur on the mass shell

$$p_\alpha p^\alpha = m^2 c^2. \quad (2.3)$$

Here,  $u^\alpha \equiv \gamma v^\alpha$  is the normalized four-velocity and  $\gamma = (1 + |\mathbf{u}|^2)^{1/2}$  is the relativistic factor.

Since the mass constraint (2.3) cannot be derived from the primitive Lagrangian (2.2), we explicitly introduce it by means of a Lagrange multiplier:

$$L_p \equiv -p_\alpha c v^\alpha - \frac{1}{2\lambda} (m^2 c^4 - p_\alpha p^\alpha c^2), \quad (2.4)$$

where  $\lambda^{-1}$  is the Lagrangian multiplier and the factor  $1/2$  is added for convenience. Since the Lagrangian (2.4) is independent of  $\dot{p}_\alpha$ , the Euler-Lagrange equation for  $p_\alpha$  yields

$$\frac{\partial L_p}{\partial p_\alpha} = -c v^\alpha + \frac{p^\alpha c^2}{\lambda} \equiv 0, \quad (2.5)$$

from which we obtain

$$p^\alpha = \lambda v^\alpha / c. \quad (2.6)$$

Using the mass constraint (2.3) and the identity  $v \cdot v \equiv \gamma^{-2}$ , the relation (2.6) yields

$$\lambda = \gamma m c^2, \quad (2.7)$$

i.e.,  $\lambda$  is the energy of a single particle of mass  $m$ .

If we now substitute (2.6) into the primitive Lagrangian (2.4) (i.e., by constraining the physical motion to take place on the mass shell), we find the physical Lagrangian

$$L(v; \lambda) \equiv L_p(x; p = \lambda v/c; \lambda) = -\frac{m^2 c^4}{2\lambda} - \frac{\lambda}{2\gamma^2}. \quad (2.8)$$

This Lagrangian now depends only on  $v^\alpha$  and  $\lambda$  (for a free particle, there is no space-time dependence in the Lagrangian). The Euler-Lagrange equation for  $\lambda$  now yields

$$\frac{\partial L}{\partial \lambda} = \frac{1}{2} \left( \frac{m^2 c^4}{\lambda^2} - \frac{1}{\gamma^2} \right) \equiv 0, \quad (2.9)$$

which gives (2.7). Substituting of (2.7) into (2.8) yields the standard Lagrangian (2.1).

For a massless particle, on the other hand, the condition (2.9) yields

$$\gamma^{-2} = v_\alpha v^\alpha \equiv 0, \quad (2.10)$$

which states that massless particles travel at the speed of light. Here,  $\lambda$  is still the massless particle's energy since (2.6) gives  $p^0 \equiv \lambda/c$ . For a massless particle, the single-particle Lagrangian is therefore simply given by the last term in (2.8), i.e.,

$$L(v; \lambda) \equiv -\frac{\lambda}{2} v_\alpha v^\alpha. \quad (2.11)$$

This Lagrangian appears in the bosonic part of the Lagrangian for a spinning particle [38]. The Lagrange multiplier  $\lambda^{-1}$  corresponds to the “einbein” which describes the square-root metric  $e = \sqrt{g}$  along the world line in a particular gauge where the world line is parameterized by time.

### B. Lagrangian density for a Free Ideal Fluid

We now discuss the passage from the finite-dimensional single-particle Lagrangian formalism based on (2.1) to an infinite-dimensional fluid Lagrangian formalism. To obtain a Lagrangian density for a fluid composed of such particles, we multiply (2.1) by the reference-frame density  $n$ , noting that the proper density is  $N \equiv n\gamma^{-1}$ . The Lagrangian for a cold ideal fluid is therefore

$$\mathcal{L}_0 = -mc^2 N = -mc^2 n \sqrt{v^\alpha v_\alpha} = -mc^2 \sqrt{J^\alpha J_\alpha}, \quad (2.12)$$

where  $J^\alpha = nv^\alpha = \langle \bar{\psi} \hat{\gamma}^\alpha \psi / c \rangle$  is the flux four-vector with a suitable ensemble average  $\langle \cdots \rangle$ . The Lorentz invariance is manifest in the last expression.

Another contribution to the Lagrangian density of an ideal fluid is the term  $-N\epsilon(N, S)$  associated with the internal energy density of the fluid in its rest frame, where the internal energy  $\epsilon(N, S)$  is a function of the proper fluid density  $N$  and its entropy  $S$  (a Lorentz scalar). By combining these two terms, the Lagrangian density for a free relativistic fluid is therefore written as

$$\mathcal{L}_0 = -N \left[ mc^2 + \epsilon(N, S) \right] \equiv -N \varepsilon(N, S), \quad (2.13)$$

where the total internal energy

$$\varepsilon(N, S) \equiv mc^2 + \epsilon(N, S) \quad (2.14)$$

includes the particle's rest energy.

As discussed above, the single-particle Lagrangian for a free massless particle is given as (2.11). The Lagrangian density for a cold ideal fluid composed of massless neutrinos is therefore given as

$$\mathcal{L}_0 \equiv -\frac{\lambda'_\nu}{2} J_\nu \cdot J_\nu = -\frac{n_\nu \lambda'_\nu}{2} v_\nu \cdot v_\nu, \quad (2.15)$$

where  $\lambda'_\nu$  is a Lorentz-scalar Lagrange multiplier field. The last expression is equivalent upon changing the variable  $\lambda_\nu = n_\nu \lambda'_\nu$ .

### III. CONSTRAINED VARIATIONAL PRINCIPLE

The self-consistent nonlinear neutrino-plasma fluid equations presented in this paper are derived from the variational principle:

$$\delta \int d^4x \mathcal{L} (A^\alpha, F^{\alpha\beta}; N_s, u_s^\alpha, S_s) = 0, \quad (3.1)$$

where in addition to its dependence on the electromagnetic four-potential  $A^\alpha$  and the Faraday tensor  $F^{\alpha\beta}$ , the Lagrangian density  $\mathcal{L}$  depends on the proper density  $N_s \equiv n_s \gamma_s^{-1}$ , the normalized fluid four-velocity  $u_s^\alpha \equiv (\gamma_s, \mathbf{u}_s)$ , and the proper internal energy (per particle)  $\varepsilon_s$  for each fluid species  $s$  (here,  $s = \sigma$  denotes a plasma-fluid species and  $s = \nu$  denotes a neutrino-fluid species).

The proper internal energy  $\varepsilon_s(N_s, S_s)$  includes the particle's rest energy [see Eq. (2.13)] and depends on the proper density  $N_s$  and the entropy  $S_s$  (a Lorentz scalar). The first law of thermodynamics [39–41] is written as

$$d\varepsilon_s = T_s dS_s - p_s dN_s^{-1}, \quad (3.2)$$

where  $T_s$  is the proper temperature and  $p_s$  is the scalar pressure for fluid species  $s$ . In what follows we use the chemical potential for each fluid species  $s$ :

$$\mu_s \equiv \partial(\varepsilon_s N_s) / \partial N_s = \varepsilon_s + p_s / N_s, \quad (3.3)$$

which represents the total energy required to create a particle of species  $s$  and inject it in a fluid sample composed of particles of the same species. Associated with the definition for the chemical potential (3.3), we also use the identity

$$\partial^\alpha \mu_s = T_s \partial^\alpha S_s + N_s^{-1} \partial^\alpha p_s. \quad (3.4)$$

Note that the independent fluid variables for each fluid species are  $N_s$ ,  $u_s^\alpha$  and  $S_s$  although other combinations are possible [32].

The Lagrangian formulation for the nonlinear interaction between neutrino and plasma fluids in the presence of an electromagnetic field is expressed in terms of the Lagrangian density

$$\mathcal{L} = - \sum_{s=\sigma, \nu} N_s \varepsilon_s - \sum_{\sigma} J_{\sigma} \cdot \left( q_{\sigma} A + \sum_{\nu} G_{\sigma\nu} J_{\nu} \right) + \frac{1}{16\pi} \mathbf{F} : \mathbf{F}, \quad (3.5)$$

where  $\mathbf{F} : \mathbf{F} \equiv F^{\alpha\beta} F_{\beta\alpha}$ . The first term in (3.5) denotes the total internal energy density of fluid  $s$ . The second term denotes the standard coupling between a charged (plasma) fluid and an electromagnetic field. The third term denotes the coupling between the neutrino-fluid species  $\nu$  and the plasma-fluid species  $\sigma$ . Note that the second and third terms can be written as  $\sum_{\sigma} n_{\sigma} V_{\sigma}$ , where the single-particle velocity  $\mathbf{v}$  in (1.8) is replaced with the fluid velocity  $\mathbf{v}_{\sigma}$ . The fourth term is the familiar electromagnetic field Lagrangian.

In the variational principle (3.1), the variation  $\delta\mathcal{L}$  is explicitly written as

$$\delta\mathcal{L} \equiv \delta A \cdot \frac{\partial\mathcal{L}}{\partial A} - \delta\mathbf{F} : \frac{\partial\mathcal{L}}{\partial\mathbf{F}} + \sum_s \left( \delta N_s \frac{\partial\mathcal{L}}{\partial N_s} + \delta u_s \cdot \frac{\partial\mathcal{L}}{\partial u_s} + \delta S_s \frac{\partial\mathcal{L}}{\partial S_s} \right), \quad (3.6)$$



where  $\delta F_{\alpha\beta} = \partial_\alpha \delta A_\beta - \partial_\beta \delta A_\alpha$  so that the second term in (3.6) can also be written as  $+2\partial\delta A : \partial\mathcal{L}/\partial\mathbf{F}$ . In contrast to other variational principles [32,42], the Eulerian variations  $\delta N_s$ ,  $\delta u_s$  and  $\delta S_s$  in (3.6) are not arbitrary but are instead *constrained*.

To obtain the correct Eulerian variation, recall that the variation of the fluid motion is an infinitesimal displacement of the fluid elements. With a fluid element  $s$  described by the four-coordinate  $x_s^\alpha$ , the normalized velocity four-vector is given by

$$u_s^\alpha(x) = \frac{dx_s^\alpha}{d\tau} \left( \frac{dx_s}{d\tau} \cdot \frac{dx_s}{d\tau} \right)^{-1/2} \equiv \left| \frac{dx_s}{d\tau} \right|^{-1} \frac{dx_s^\alpha}{d\tau}, \quad (3.7)$$

where  $\tau$  parametrizes the world line of the fluid element. Under the infinitesimal displacement  $x_s^\alpha \rightarrow x_s^\alpha + \delta\xi_s^\alpha$  [with  $\delta\xi_s^\alpha \equiv (\delta\xi_s^0, \delta\boldsymbol{\xi}_s)$ ], the apparent variation at a position following a fluid element along its worldline is

$$\begin{aligned} & \frac{d\delta\xi_s^\alpha}{d\tau} \left| \frac{dx_s}{d\tau} \right|^{-1} - \frac{dx_s^\alpha}{d\tau} \left| \frac{dx_s}{d\tau} \right|^{-3} \frac{d\delta\xi_s}{d\tau} \cdot \frac{dx_s}{d\tau} \\ &= (u_s \cdot \partial) \delta\xi_s^\alpha - u_s^\alpha [u_{s\beta} (u_s \cdot \partial) \delta\xi_s^\beta] \equiv h_s^{\alpha\beta} (u_s \cdot \partial) \delta\xi_{s\beta}, \end{aligned} \quad (3.8)$$

with  $u_s \cdot \partial \equiv |dx_s/d\tau|^{-1} d/d\tau$  and

$$h_s^{\alpha\beta} \equiv g^{\alpha\beta} - u_s^\alpha u_s^\beta \quad (3.9)$$

is a symmetric projection tensor [40] (i.e.,  $h_s \cdot u_s \equiv 0$ ). The Eulerian variation at a fixed space-time location is therefore given by [33]

$$\delta u_s^\alpha(x) = h_s^{\alpha\beta} (u_s \cdot \partial) \delta\xi_{s\beta} - (\delta\xi_s \cdot \partial) u_s^\alpha. \quad (3.10)$$

It is easy to check that this variation preserves  $u^\alpha u_\alpha = 1$ .

The variation of the proper density  $N_s$  can be obtained by the requirement that the quantity

$$N_s \left( \frac{dx_s}{d\tau} \cdot \frac{dx_s}{d\tau} \right)^{-1/2} d^4x \quad (3.11)$$

should be kept intact (i.e., mass is conserved). The factor in the bracket is the induced metric along the world line. This requirement fixes the variation at a position following a fluid element along its worldline as  $-N_s[(\partial \cdot \delta\xi_s) - u_{s\beta}(u_s \cdot \partial)\delta\xi_s^\beta] = -N_s[h_s^{\alpha\beta}\partial_\alpha\delta\xi_{s\beta}]$ , and hence the Eulerian variation is given by

$$\delta N_s = -(\delta\xi_s \cdot \partial) N_s - N_s h_s : \partial\delta\xi_s. \quad (3.12)$$

It is straightforward to check that the above variations Eqs. (3.10, 3.12) are consistent with the conservation law

$$\partial_\alpha J_s^\alpha = 0 \quad (3.13)$$

of the flux four-vector  $J_s^\alpha = N_s u_s^\alpha$ . It is useful to know its variation which can be easily calculated using Eqs. (3.10, 3.12):

$$\delta J_s^\alpha = \partial_\beta (J_s^\beta \delta \xi_s^\alpha - J_s^\alpha \delta \xi_s^\beta), \quad (3.14)$$

where the conservation law (3.13) has been used.

Finally, the non-dissipative flow conserves entropy along the world line,

$$(u_s \cdot \partial) S_s = 0. \quad (3.15)$$

To be consistent with the variation Eq. (3.10), we find

$$\delta S_s = -(\delta \xi_s \cdot \partial) S_s. \quad (3.16)$$

The expressions (3.10, 3.12, 3.16) give the correct relativistic generalizations of the (non-relativistic) constrained Eulerian variations [43]; see Appendix A for a geometric interpretation of Eqs. (3.10, 3.12, 3.14, 3.16). An alternative variational principle would introduce  $\partial \cdot J_s = 0 = u_s \cdot \partial S_s$  explicitly in the Lagrangian density by means of Lagrange multipliers [32].

#### IV. SELF-CONSISTENT NONLINEAR NEUTRINO-PLASMA FLUID EQUATIONS

We now proceed with the variational derivation of the dynamical equations for self-consistent neutrino-plasma fluid interactions. In deriving these equations, we use the thermodynamic relations (3.2)-(3.4) as well as the continuity and entropy equations (3.13, 3.15) for each fluid species  $s$ .

By re-arranging terms in the variational equation (3.6) so as to isolate the variation four-vectors  $\delta \xi_s$  and  $\delta A$ , we find

$$\begin{aligned} \delta \mathcal{L} \equiv \partial \cdot \mathcal{J} - \sum_s \delta \xi_s \cdot \left[ \partial_s \mathcal{L} + \partial \cdot \left( u_s \frac{\partial \mathcal{L}}{\partial u_s} \cdot \mathbf{h}_s - N_s \frac{\partial \mathcal{L}}{\partial N_s} \mathbf{h}_s \right) \right] \\ + \delta A \cdot \left( \frac{\partial \mathcal{L}}{\partial A} - 2 \partial \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \right), \end{aligned} \quad (4.1)$$

where  $\partial_s \mathcal{L} \equiv \partial N_s (\partial \mathcal{L} / \partial N_s) + \partial u_s \cdot (\partial \mathcal{L} / \partial u_s) + \partial S_s (\partial \mathcal{L} / \partial S_s)$ , and the Noether four-density  $\mathcal{J}$  is expressed in terms of  $\delta \xi_s$  and  $\delta A$  as

$$\mathcal{J} \equiv \sum_s \left( u_s \frac{\partial \mathcal{L}}{\partial u_s} \cdot \mathbf{h}_s - N_s \frac{\partial \mathcal{L}}{\partial N_s} \mathbf{h}_s \right) \cdot \delta \xi_s + 2 \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \cdot \delta A. \quad (4.2)$$

When performing the variational principle (3.1), with  $\delta \mathcal{L}$  given by (4.1), we only consider variations  $\delta \xi_s$  and  $\delta A$  which vanish on the integration boundary. Hence, the Noether density  $\mathcal{J}$  in (4.1) does not contribute to the dynamical equations.

##### A. Plasma-Fluid Momentum Equation

First, we derive the relativistic plasma-fluid four-momentum equation. Upon variation with respect to  $\delta \xi_\sigma$  in (3.1), we obtain

$$0 = \left( \partial N_\sigma \frac{\partial \mathcal{L}}{\partial N_\sigma} + \partial u_\sigma \cdot \frac{\partial \mathcal{L}}{\partial u_\sigma} + \partial S_\sigma \frac{\partial \mathcal{L}}{\partial S_\sigma} \right) + \partial \cdot \left( u_\sigma \frac{\partial \mathcal{L}}{\partial u_\sigma} \cdot \mathbf{h}_\sigma - N_\sigma \frac{\partial \mathcal{L}}{\partial N_\sigma} \mathbf{h}_\sigma \right). \quad (4.3)$$

Substitution of appropriate derivatives of the Lagrangian density  $\mathcal{L}$  and using the constraint equations (3.13, 3.15) and the thermodynamic relations (3.2)-(3.4), this equation becomes the relativistic plasma-fluid four-momentum (covariant) equation

$$u_\sigma \cdot \partial (\mu_\sigma u_\sigma) = N_\sigma^{-1} \partial p_\sigma + \left( q_\sigma \mathbf{F} + \sum_\nu G_{\sigma\nu} \mathbf{M}_\nu \right) \cdot u_\sigma, \quad (4.4)$$

where

$$M_\nu^{\alpha\beta} \equiv \partial^\alpha J_\nu^\beta - \partial^\beta J_\nu^\alpha \quad (4.5)$$

is an anti-symmetric tensor which represents the influence of the neutrino background medium [44]. This tensor satisfies the Maxwell-like equation  $\partial^\rho M_\nu^{\alpha\beta} + \partial^\alpha M_\nu^{\beta\rho} + \partial^\beta M_\nu^{\rho\alpha} \equiv 0$  and its divergence is  $\partial_\alpha M_\nu^{\alpha\beta} \equiv \square J_\nu^\beta$ , where  $\square \equiv \partial \cdot \partial$  and the continuity equation  $\partial \cdot J_\nu = 0$  for the neutrino fluid was used.

Separating the space and time components in (4.4) (i.e., using the 3 + 1 notation), the spatial components of the plasma-fluid four-momentum equation (4.4) yield

$$(\partial_t + \mathbf{v}_\sigma \cdot \nabla) (\mu_\sigma \gamma_\sigma \mathbf{v}_\sigma / c^2) = -n_\sigma^{-1} \nabla p_\sigma + q_\sigma \left( \mathbf{E} + \frac{\mathbf{v}_\sigma}{c} \times \mathbf{B} \right) + \mathbf{f}_\sigma, \quad (4.6)$$

where  $\mathbf{f}_\sigma$  is the neutrino-induced ponderomotive force (averaged over neutrino species) on the plasma-fluid species  $\sigma$ , defined as

$$\mathbf{f}_\sigma \equiv \sum_\nu G_{\sigma\nu} \left[ - \left( \nabla n_\nu + \frac{1}{c} \frac{\partial \mathbf{J}_\nu}{\partial t} \right) + \frac{\mathbf{v}_\sigma}{c} \times \nabla \times \mathbf{J}_\nu \right]. \quad (4.7)$$

The neutrino-induced ponderomotive force  $\mathbf{f}_\sigma$  is composed of three terms: an electrostatic-like term  $\nabla n_\nu$ , an inductive-like term  $\partial_t \mathbf{J}_\nu$ , and a magnetic-like term  $\nabla \times \mathbf{J}_\nu$ . This terminology is obviously motivated by the similarities with the electromagnetic force on a charged particle. In previous work by Silva *et al.* [17], only the electrostatic-like term is retained in the neutrino-induced ponderomotive force, i.e., the neutrino particle flux  $\mathbf{J}_\nu$  is discarded under the assumption of isotropic neutrino and plasma fluids.

## B. Neutrino-Fluid Momentum Equation

Next, we derive the relativistic neutrino-fluid four-momentum equation; the limiting case of zero neutrino masses is treated below (4.12). Upon variation with respect to  $\delta \xi_\nu$  in (3.1), we obtain

$$0 = \left( \partial N_\nu \frac{\partial \mathcal{L}}{\partial N_\nu} + \partial u_\nu \cdot \frac{\partial \mathcal{L}}{\partial u_\nu} + \partial S_\nu \frac{\partial \mathcal{L}}{\partial S_\nu} \right) + \partial \cdot \left( u_\nu \frac{\partial \mathcal{L}}{\partial u_\nu} \cdot \mathbf{h}_\nu - N_\nu \frac{\partial \mathcal{L}}{\partial N_\nu} \mathbf{h}_\nu \right). \quad (4.8)$$

Substitution of derivatives of  $\mathcal{L}$  and using the thermodynamic relations (3.2)-(3.4), this equation becomes the relativistic neutrino-fluid four-momentum equation

$$u_\nu \cdot \partial (\mu_\nu u_\nu) = N_\nu^{-1} \partial p_\nu + \sum_\sigma G_{\sigma\nu} \mathbf{M}_\sigma \cdot u_\nu, \quad (4.9)$$

where

$$M_\sigma^{\alpha\beta} \equiv \partial^\alpha J_\sigma^\beta - \partial^\beta J_\sigma^\alpha \quad (4.10)$$

is another anti-symmetric tensor which represents the influence of the background medium. This tensor satisfies the Maxwell-like equation  $\partial^\rho M_\sigma^{\alpha\beta} + \partial^\alpha M_\sigma^{\beta\rho} + \partial^\beta M_\sigma^{\rho\alpha} \equiv 0$  and its divergence is  $\partial_\alpha M_\sigma^{\alpha\beta} \equiv \square J_\sigma^\beta$ , where the continuity equation  $\partial \cdot J_\sigma = 0$  for the plasma fluid was used. In (4.9), we note that the neutrino fluid is thus under the influence of an electromagnetic-like force induced by nonuniform plasma flows. We also note that the symmetry between the ponderomotive forces (4.5) and (4.10) is a result of the symmetry of the neutrino-plasma interaction term ( $\sum_\sigma \sum_\nu G_{\sigma\nu} J_\sigma \cdot J_\nu$ ) in the Lagrangian density (3.5).

Using the 3+1 notation, the spatial components of neutrino-fluid four-momentum equation (4.9) yield

$$(\partial_t + \mathbf{v}_\nu \cdot \nabla) (\mu_\nu \gamma_\nu \mathbf{v}_\nu / c^2) = -n_\nu^{-1} \nabla p_\nu + \mathbf{f}_\nu, \quad (4.11)$$

where  $\mathbf{f}_\nu$  is the plasma-induced ponderomotive force (averaged over plasma-particle species) on the neutrino-fluid species  $\nu$ , defined as

$$\mathbf{f}_\nu \equiv \sum_\sigma G_{\sigma\nu} \left[ - \left( \nabla n_\sigma + \frac{1}{c} \frac{\partial \mathbf{J}_\sigma}{\partial t} \right) + \frac{\mathbf{v}_\nu}{c} \times \nabla \times \mathbf{J}_\sigma \right]. \quad (4.12)$$

The plasma-induced ponderomotive force  $\mathbf{f}_\nu$  on the neutrino fluid is composed of three terms: an electrostatic-like term  $\nabla n_\sigma$ , an inductive-like term  $\partial_t \mathbf{J}_\sigma$ , and a magnetic-like term  $\nabla \times \mathbf{J}_\sigma$ .

We now discuss the case of a cold ideal fluid composed of massless neutrinos. Variation of the neutrino part of the Lagrangian density

$$\mathcal{L}_\nu \equiv -\frac{1}{2} n_\nu \lambda_\nu v_\nu \cdot v_\nu - \sum_\sigma G_{\sigma\nu} J_\sigma \cdot J_\nu$$

with respect to  $\delta \xi_\nu$  yields

$$\delta \mathcal{L}_\nu \equiv -n_\nu \lambda_\nu \delta v_\nu \cdot v_\nu - \sum_\sigma G_{\sigma\nu} J_\sigma \cdot \delta J_\nu, \quad (4.13)$$

where we used the constraint  $v_\nu \cdot v_\nu \equiv 0$ . Using  $\delta J_\nu \equiv \partial \cdot (J_\nu \delta \xi_\nu - \delta \xi_\nu J_\nu)$  and  $\delta v_\nu \cdot J_\nu = \partial \cdot (J_\nu \delta \xi_\nu) \cdot v_\nu$ , the variation equation (4.13) becomes

$$\delta \mathcal{L}_\nu \equiv \partial \cdot \mathcal{J} + n_\nu \delta \xi_\nu \cdot \left[ v_\nu \cdot \partial (\lambda_\nu v_\nu) - \sum_\sigma G_{\sigma\nu} \mathbf{M}_\sigma \cdot v_\nu \right], \quad (4.14)$$

where the tensor  $\mathbf{M}_\sigma$  is defined in (4.10) and the Noether density is

$$\mathcal{J} \equiv \delta \xi_\nu \cdot \left[ \mathbf{g} G_{\sigma\nu} J_\nu \cdot J_\sigma - J_\nu \left( \lambda_\nu v_\nu + \sum_\sigma G_{\sigma\nu} J_\sigma \right) \right]. \quad (4.15)$$

From (4.14) the variational principle  $\int \delta \mathcal{L}_\nu d^4x = 0$  yields the cold neutrino fluid equation

$$v_\nu \cdot \partial(\lambda_\nu v_\nu) = \sum_\sigma G_{\sigma\nu} \mathbf{M}_\sigma \cdot v_\nu. \quad (4.16)$$

In the cold-fluid limit, on the other hand, (4.9) yields  $u_\nu \cdot \partial(\gamma_\nu^{-1} \lambda_\nu u_\nu) = \sum_\sigma G_{\sigma\nu} \mathbf{M}_\sigma \cdot u_\nu$ , where  $\lambda_\nu$  is the neutrino energy. By substituting  $u_\nu \equiv \gamma_\nu v_\nu$  into this expression, we readily check that (4.9) and (4.16) are identical in the massless-neutrino cold-fluid limit and that (4.9) can in fact be used to describe neutrino-fluid dynamics with arbitrary neutrino mass.

### C. Maxwell Equations

The remaining equations are obtained from the variational principle (3.1) upon variations with respect to the four-potential  $\delta A^\alpha$ . One thus obtains

$$0 = \frac{\partial \mathcal{L}}{\partial A} - 2 \partial \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{F}}. \quad (4.17)$$

Substitution of derivatives of  $\mathcal{L}$ , this equation becomes the Maxwell equation

$$\partial \cdot \mathbf{F} = 4\pi \sum_\sigma q_\sigma J_\sigma. \quad (4.18)$$

Using the  $3+1$  notation, we recover one half of the familiar Maxwell equations from (4.18). The other half is expressed in terms of the Faraday tensor alone as

$$\partial^\rho F^{\alpha\beta} + \partial^\alpha F^{\beta\rho} + \partial^\beta F^{\rho\alpha} \equiv 0, \quad (4.19)$$

which, using the  $3+1$  notation, yields  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{E} + c^{-1} \partial_t \mathbf{B} = 0$ .

### D. Energy-Momentum Conservation Laws

Since the dynamical equations (4.3), (4.8) and (4.17) are true for arbitrary variations  $(\delta \xi_\sigma, \delta \xi_\nu)$  and  $\delta A$  (subject to boundary conditions), the variational equation (4.1) becomes

$$\delta \mathcal{L} \equiv \partial \cdot \mathcal{J}, \quad (4.20)$$

which we henceforth refer to as the Noether equation. We now discuss Noether symmetries of the Lagrangian density (3.5) based on the Noether equation (4.20).

For this purpose, we consider infinitesimal translations  $x^\alpha \rightarrow x^\alpha + \delta x^\alpha$  generated by the infinitesimal displacement four-vector field  $\delta x$ . Under this transformation, the Lagrangian density  $\mathcal{L}$  changes by

$$\delta \mathcal{L} \equiv -\partial \cdot (\delta x \mathcal{L}). \quad (4.21)$$

Next, we introduce the following explicit expressions for  $(\delta \xi_\sigma, \delta \xi_\nu)$  and  $\delta A$  in terms of the infinitesimal generating four-vector  $\delta x$ :

$$\left. \begin{aligned} \delta \xi_s &\equiv \mathbf{h}_s \cdot \delta x \\ \delta A &\equiv \mathbf{F} \cdot \delta x - \partial(A \cdot \delta x) \end{aligned} \right\}, \quad (4.22)$$

where the symmetric tensor  $\mathbf{h}_s$  is defined in (3.9). (These expressions are given geometric interpretations in Appendix A.)

Substituting (4.22) in the Noether density (4.2), we find

$$\mathcal{J} = \left[ 2 \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \cdot \mathbf{F} + \sum_s \left( u_s \frac{\partial \mathcal{L}}{\partial u_s} \cdot \mathbf{h}_s - N_s \frac{\partial \mathcal{L}}{\partial N_s} \mathbf{h}_s \right) \right] \cdot \delta x + 2 \partial(A \cdot \delta x) \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{F}}, \quad (4.23)$$

where we have used the identity  $\mathbf{h}_s \cdot \mathbf{h}_s = \mathbf{h}_s$  in writing the second and third terms. Making use of the Maxwell equation (4.18), the last term in (4.23) can be re-arranged as

$$2 \partial(A \cdot \delta x) \cdot \frac{\partial \mathcal{L}}{\partial \mathbf{F}} = \partial \cdot \left[ 2(A \cdot \delta x) \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \right] - (A \cdot \delta x) \frac{\partial \mathcal{L}}{\partial A}. \quad (4.24)$$

We now note that the expression for  $\partial \cdot \mathcal{J}$  in (4.20) is invariant under the transformation  $\mathcal{J} \rightarrow \mathcal{J} + \partial \cdot \mathbf{K}$ , where  $\mathbf{K}$  is an antisymmetric tensor (for which  $\partial_{\alpha\beta}^2 K^{\alpha\beta} \equiv 0$ ) which vanishes on the integration boundary in (3.1). Since the first term on the right side of (4.24) is such a term, it can be transformed away and the final expression for the Noether density is therefore

$$\mathcal{J} = \left[ 2 \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \cdot \mathbf{F} - \frac{\partial \mathcal{L}}{\partial A} A + \sum_s \left( u_s \frac{\partial \mathcal{L}}{\partial u_s} \cdot \mathbf{h}_s - N_s \frac{\partial \mathcal{L}}{\partial N_s} \mathbf{h}_s \right) \right] \cdot \delta x. \quad (4.25)$$

Substituting (4.21) into (4.20), the Noether equation becomes  $\partial \cdot (\mathcal{J} + \delta x \mathcal{L}) = 0$ . We define the symmetric energy-momentum tensor  $\mathbf{T}$  from the expression

$$\mathcal{J} + \delta x \mathcal{L} \equiv -\mathbf{T} \cdot \delta x, \quad (4.26)$$

where, using (4.25), the energy-momentum tensor  $\mathbf{T}$  is explicitly given as

$$\mathbf{T} = -\mathbf{g} \mathcal{L} - \left( 2 \frac{\partial \mathcal{L}}{\partial \mathbf{F}} \cdot \mathbf{F} - \frac{\partial \mathcal{L}}{\partial A} A \right) - \sum_s \left( u_s \frac{\partial \mathcal{L}}{\partial u_s} \cdot \mathbf{h}_s - N_s \frac{\partial \mathcal{L}}{\partial N_s} \mathbf{h}_s \right). \quad (4.27)$$

For a constant translation  $\delta x$ , the Noether equation (4.20) then becomes

$$0 = \partial \cdot \mathbf{T}, \quad (4.28)$$

where using the Lagrangian density (3.5) and its derivatives in (4.27), we find

$$\begin{aligned} T^{\alpha\beta} &= \frac{1}{4\pi} \left( F^\alpha_\kappa F^{\kappa\beta} - \frac{g^{\alpha\beta}}{4} \mathbf{F} : \mathbf{F} \right) + \sum_s \left( N_s \mu_s u_s^\alpha u_s^\beta - p_s g^{\alpha\beta} \right) \\ &\quad + \sum_\sigma \sum_\nu G_{\sigma\nu} \left( J_\sigma^\alpha J_\nu^\beta + J_\nu^\alpha J_\sigma^\beta - g^{\alpha\beta} J_\sigma \cdot J_\nu \right). \end{aligned} \quad (4.29)$$

This energy-momentum tensor contains the usual terms associated with an electromagnetic field and a free relativistic ideal fluid [39–41]. It also contains the energy-momentum terms associated with collective neutrino-plasma interactions (third set of terms).

The energy-momentum transfer between the electromagnetic-plasma background and the neutrinos can now be investigated. Such a process is relevant to supernova explosions, for example, where approximately 1% of the neutrino energy needs to be transferred to the surrounding plasma. First, we define the electromagnetic-plasma (EMP) energy-momentum tensor:

$$\mathbf{T}_{\text{EMP}} \equiv \frac{1}{4\pi} \left( \mathbf{F} \cdot \mathbf{F} - \frac{\mathbf{g}}{4} \mathbf{F} : \mathbf{F} \right) + \sum_{\sigma} (N_{\sigma} \mu_{\sigma} u_{\sigma} u_{\sigma} - p_{\sigma} \mathbf{g}) \equiv \mathbf{T}_{\text{EM}} + \mathbf{T}_{\text{P}}, \quad (4.30)$$

and, using the exact energy-momentum conservation law (4.28) as well as the dynamical equations (4.4), (4.9) and (4.18), we find

$$\partial \cdot \mathbf{T}_{\text{EMP}} = \sum_{\sigma} \left( \sum_{\nu} G_{\sigma\nu} \mathbf{M}_{\nu} \right) \cdot J_{\sigma}. \quad (4.31)$$

This equation describes how energy and momentum are transferred from the neutrinos to the electromagnetic field and the background plasma. Note how the transfer of energy-momentum between an electromagnetic-plasma and neutrinos is very much like the transfer of energy between a plasma (P) and an electromagnetic field (i.e.,  $\partial \cdot \mathbf{T}_{\text{P}} = \sum_{\sigma} q_{\sigma} \mathbf{F} \cdot J_{\sigma}$ ) in the absence of neutrinos.

We note that in addition to energy and momentum, wave action [15] can be transferred between the neutrinos and the electromagnetic-plasma background. In this case, electromagnetic waves and/or plasma waves can be excited by resonant three-wave processes.

## V. MAGNETIC FIELD GENERATION AND HELICITY PRODUCTION BY COLLECTIVE NEUTRINO-PLASMA INTERACTIONS

An important application of the process of energy-momentum transfer associated with collective electromagnetic-plasma-neutrino interactions is the possibility of generating magnetic fields in an unmagnetized plasma as a result of collective neutrino-plasma interactions. Such process might be relevant to the problem of magnetogenesis and the production of magnetic helicity in the early universe [45–47]. A similar process of magnetic-field generation has been observed in laser-plasma interactions [26–29].

According to our neutrino-plasma fluid model [based on (4.4), (4.9), and (4.18)], the strength of the magnetic field generated by neutrino-plasma interactions scales as the first power in the Fermi weak-interaction constant  $G_F$ . In what follows, we thus refer to magnetic fields generated by classical plasma processes (e.g., the Biermann-battery effect and the nonlinear dynamo effect) as zeroth-order fields while those generated by collective neutrino-plasma interactions as first-order fields. Second-order fields, for example, might be produced by processes such as  $\sigma' \rightarrow \nu \rightarrow \sigma \rightarrow EM$ , where the first plasma-particle species ( $\sigma'$ ) need not be charged (e.g., neutrons).

In this Section, we investigate the role played by collective neutrino-plasma interactions in generating magnetic fields and magnetic helicity as well as magnetic equilibrium.

### A. Magnetic-field Generation

An equation describing magnetic-field generation resulting from collective neutrino-plasma interactions is derived as follows. We begin with Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}, \quad (5.1)$$

where for a given plasma-particle species  $\sigma$  [using (4.7)], the electric field  $\mathbf{E}$  is expressed as

$$\mathbf{E} \equiv \frac{1}{q_\sigma} (\mathbf{F}_\sigma - \mathbf{f}_\sigma) - \frac{\mathbf{v}_\sigma}{c} \times \mathbf{B}, \quad (5.2)$$

where  $\mathbf{f}_\sigma$  is the neutrino-induced ponderomotive force given by (4.7) and

$$\mathbf{F}_\sigma \equiv \frac{\partial \mathbf{P}_\sigma}{\partial t} + \mathbf{v}_\sigma \cdot \nabla \mathbf{P}_\sigma + n_\sigma^{-1} \nabla p_\sigma, \quad (5.3)$$

with  $\mathbf{P}_\sigma \equiv (\mu_\sigma/c^2) \gamma_\sigma \mathbf{v}_\sigma$  the generalized momentum for plasma-fluid species  $\sigma$ .

Since the electric field  $\mathbf{E}$  is common to all charged-particle species, we multiply (5.2) on both sides by  $q_\sigma^2$  and sum over all charged-particle species present in the plasma. Defining  $\sum_\sigma q_\sigma^2 \equiv Q^2$ , the electric field  $\mathbf{E}$  is then given as

$$\mathbf{E} = \sum_\sigma \frac{q_\sigma}{Q^2} (\mathbf{F}_\sigma - \mathbf{f}_\sigma) - \left( \sum_\sigma \frac{q_\sigma^2 \mathbf{v}_\sigma}{c Q^2} \right) \times \mathbf{B}. \quad (5.4)$$

Substituting explicit expressions for  $\mathbf{F}_\sigma$  and  $\mathbf{f}_\sigma$ , we obtain

$$\mathbf{E} \equiv \sum_\sigma \frac{q_\sigma}{Q^2} \left[ \partial_t \mathbf{\Pi}_\sigma - \mathbf{v}_\sigma \times \nabla \times \mathbf{\Pi}_\sigma + \nabla \chi_\sigma + S_\sigma \nabla (\gamma_\sigma^{-1} T_\sigma) \right] - \left( \sum_\sigma \frac{q_\sigma^2 \mathbf{v}_\sigma}{c Q^2} \right) \times \mathbf{B}, \quad (5.5)$$

where  $\gamma_\sigma^{-1} T_\sigma$  is the temperature in the lab reference frame and

$$\left. \begin{aligned} \mathbf{\Pi}_\sigma &\equiv \mathbf{P}_\sigma + \sum_\nu G_{\sigma\nu} \mathbf{J}_\nu / c \\ \chi_\sigma &\equiv \sum_\nu G_{\sigma\nu} n_\nu + \gamma_\sigma \mu_\sigma - \gamma_\sigma^{-1} T_\sigma S_\sigma \end{aligned} \right\}. \quad (5.6)$$

Eq. (5.5) can then be substituted for the electric field into Faraday's law (5.1) to give

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \sum_\sigma \frac{c q_\sigma}{Q^2} \left[ \nabla (\gamma_\sigma^{-1} T_\sigma) \times \nabla S_\sigma - \nabla \times (\partial_t \mathbf{P}_\sigma - \mathbf{v}_\sigma \times \nabla \times \mathbf{P}_\sigma) \right] \\ &+ \sum_\sigma \frac{q_\sigma^2}{Q^2} \nabla \times (\mathbf{v}_\sigma \times \mathbf{B}) - \sum_\nu \sum_\sigma \frac{q_\sigma G_{\sigma\nu}}{Q^2} \nabla \times (\partial_t \mathbf{J}_\nu - \mathbf{v}_\sigma \times \nabla \times \mathbf{J}_\nu). \end{aligned} \quad (5.7)$$

The first collection of terms (linear in  $q_\sigma$ ) on the right side of (5.7) includes the so-called Biermann-battery term ( $\nabla n_\sigma^{-1} \times \nabla T_\sigma$ ) [27,28,48] while the second term (proportional to  $q_\sigma^2$ ) represents the nonlinear dynamo effect. These classical (zeroth-order) terms have been known to play important roles in the generation of magnetic fields during laser-plasma interactions [26–29] as well as the evolution of cosmic and galactic magnetic fields [48].



The last collection of terms (proportional to  $q_\sigma G_{\sigma\nu}$ ) in (5.7) are associated with collective neutrino-plasma interactions and are completely new. Here, the neutrino-flux vorticity ( $\nabla \times \mathbf{J}_\nu$ ) plays a fundamental role in generating first-order magnetic fields; such terms are completely missing from previous works [30,31].

According to (5.7), the electrostatic part of the neutrino-induced ponderomotive force (4.7) does not play any role in generating magnetic fields. Indeed, for each neutrino-fluid species  $\nu$ , we have  $\nabla \times [(\sum_s q_s G_{s\nu}) \nabla n_\nu] = 0$ , independent of the plasma-fluid composition. The neutrino-induced ponderomotive force on plasma particles of species  $\sigma$  actually given in [14,30,31] is  $-n_\sigma^{-1} (\sum_{s'} G_{s'\nu} n_{s'}) \nabla n_\nu \equiv \mathbf{f}_\sigma^{(B)}$ ; this expression improperly involves a sum of plasma-particle species ( $\sum_{s'}$ ) instead of the sum over neutrino species ( $\sum_\nu$ ) as it appears in (4.7). Shukla *et al.* [31] then go on to develop a model for magnetic-field generation based on the fact that  $\nabla \times \mathbf{f}_\sigma^{(B)} \neq 0$  for a plasma with multiple particles species. Since the sum over plasma-particle species ( $\sum_{s'}$ ) appearing in  $\mathbf{f}_\sigma^{(B)}$  is inappropriate, however, the conclusion drawn by Shukla *et al.* [31] that magnetic fields can be generated in a plasma composed of neutrons ( $\sigma = n$ ) and electrons ( $\sigma = e$ ) by terms such as  $\nabla(n_n/n_e) \times \nabla n_\nu$  is incorrect [49].

For a primordial plasma, we note that the Biermann-battery term could be small unless the terms  $\nabla(\gamma_\sigma^{-1} T_\sigma) \times \nabla S_\sigma$  and  $\nabla(\gamma_{\bar{\sigma}}^{-1} T_{\bar{\sigma}}) \times \nabla S_{\bar{\sigma}}$  are in opposite directions whereas the nonlinear dynamo requires net plasma flow. Using the identities (1.7), on the other hand, we note that particles ( $\sigma$ ) and anti-particles ( $\bar{\sigma}$ ) of the same family ( $\sigma, \bar{\sigma}$ ) contribute equally to the generation of first-order magnetic fields in a primordial plasma since

$$\left. \begin{aligned} \sum_{s=\sigma, \bar{\sigma}} q_s G_{s\nu} &= 2 q_\sigma G_{\sigma\nu} \\ \sum_{s=\sigma, \bar{\sigma}} q_s G_{s\nu} \mathbf{v}_s &= q_\sigma G_{\sigma\nu} (\mathbf{v}_\sigma + \mathbf{v}_{\bar{\sigma}}) \end{aligned} \right\}. \quad (5.8)$$

This remark is especially relevant to the problem of magnetogenesis in the early universe. Conversely, we note from (5.7) that a time-dependent magnetic field *automatically* generates neutrino-flux vorticity  $\nabla \times \mathbf{J}_\nu$ . Hence, the usual assumption that the neutrino distribution is isotropic [17] appears to be inconsistent with first-order magnetic-field generation by first-order collective neutrino-plasma interactions.

## B. Magnetic Helicity Production

Another quantity intimately associated with magnetic-field generation is the generation of magnetic helicity

$$H \equiv \int_V \mathbf{A} \cdot \mathbf{B} d^3x, \quad (5.9)$$

where  $V$  is the three-dimensional volume which encloses the magnetic field lines; to ensure that this definition of magnetic helicity be gauge invariant, we require that  $\mathbf{B} \cdot \hat{n} = 0$ , where  $\hat{n}$  is a unit vector normal to the surface  $\partial V$ . Magnetic helicity is a measure of knottedness (or flux linkage) in the magnetic field [50]; hence a uniform magnetic field (or more generally a magnetic field which has a global representation in terms of Euler potentials  $\alpha$  and  $\beta$  as  $\mathbf{B} \equiv \nabla\alpha \times \nabla\beta$ ) has zero helicity. The production of magnetic helicity is therefore an indication that the spatial structure (and topology) of the magnetic field is becoming more

complex. It is expected that this feature in turn plays a fundamental role in the formation of large-scale structure in the universe [2].

The time evolution of the magnetic helicity (5.9) leads to the equation

$$\frac{dH}{dt} = -2c \int_V \mathbf{E} \cdot \mathbf{B} d^3x - c \int_{\partial V} (\phi \mathbf{B} + \mathbf{E} \times \mathbf{A}) \cdot \hat{n} d^2x, \quad (5.10)$$

where integration by parts was performed in obtaining the surface term. Taking the integration volume  $V$  arbitrarily large (or requiring that  $\mathbf{E}$  be parallel to  $\hat{n}$  in addition to  $\mathbf{B} \cdot \hat{n} = 0$ ), we find that the surface term vanishes and we are left only with the first term in (5.10). If we now substitute (5.5) into (5.10), we obtain

$$\frac{dH}{dt} = - \sum_{\sigma} \frac{2q_{\sigma}c}{Q^2} \int_V \mathbf{B} \cdot \left[ \partial_t \mathbf{\Pi}_{\sigma} - \mathbf{v}_{\sigma} \times \nabla \times \mathbf{\Pi}_{\sigma} + \nabla \chi_{\sigma} + S_{\sigma} \nabla(\gamma_{\sigma}^{-1} T_{\sigma}) \right] d^3x, \quad (5.11)$$

where  $\mathbf{\Pi}_{\sigma}$  and  $\chi_{\sigma}$  are defined in (5.6). Since the term  $\mathbf{B} \cdot \nabla \chi_{\sigma}$  can be written as an exact divergence, it does not contribute to the production of magnetic helicity. Furthermore, since temperature gradients along the magnetic field,  $\mathbf{B} \cdot \nabla(\gamma_{\sigma}^{-1} T_{\sigma})$ , vanish in the absence of dissipative effects the last term in (5.11) drops out. Hence, magnetic helicity production is governed by the equation

$$\frac{dH}{dt} = - \sum_{\sigma} \frac{2q_{\sigma}c}{Q^2} \int_V \mathbf{B} \cdot (\partial_t \mathbf{\Pi}_{\sigma} - \mathbf{v}_{\sigma} \times \nabla \times \mathbf{\Pi}_{\sigma}). \quad (5.12)$$

This equation states that helicity production can occur in the presence of (zeroth-order) non-trivial flows [50] and/or (first-order) nonuniform neutrino flux.

It has been pointed out that magnetic helicity plays an important role in allowing energy to be transferred from small to large scales by a process called inverse cascade. Thus neutrino-flux vorticity leads to the generation of small-scale magnetic fields, first, and then to the production of magnetic helicity. The production of magnetic helicity, on the other hand, converts the small-scale magnetic fields to large-scale magnetic fields which are expected to play a fundamental role in the problem of structure formation in the early universe. The magnetic helicity production described by (5.12) involves a multi-species fluid picture. A more standard description is based on the magnetohydrodynamic (MHD) equations in which plasma flows are averaged over particle species. Future work will proceed by deriving ideal neutrino-MHD equations.

### C. Magnetic Equilibrium in a Magnetized Plasma and Neutrino Fluid

When gravitational effects can be ignored, plasmas can be confined by magnetic fields. Such an equilibrium is established by balancing the (outward) kinetic pressure gradient with the (inward) magnetic pressure gradient. We now investigate how magnetic equilibria are modified by the presence of neutrino fluxes.

The equation for magnetic equilibrium involving magnetic fields associated with neutrino-plasma interactions can be obtained by multiplying (5.2) with  $q_{\sigma} n_{\sigma}$  and summing over the charged-particle species only. In a time-independent equilibrium ( $\partial/\partial t \equiv 0$ ) involving a

quasi-neutral plasma (where  $\sum_{\sigma} q_{\sigma} n_{\sigma} = 0$ ), a static magnetic field  $\mathbf{B}$  and time-independent neutrino fluids, we find the following equilibrium condition

$$\begin{aligned} \frac{\mathbf{J}}{c} \times \mathbf{B} = & \nabla \cdot \left[ \sum_{\sigma} (n_{\sigma} \mathbf{v}_{\sigma} \mathbf{P}_{\sigma} + \mathbf{I} p_{\sigma}) \right] + \sum_{\nu} \left[ \left( \sum_{\sigma} n_{\sigma} G_{\sigma\nu} \right) \nabla n_{\nu} \right] \\ & - \sum_{\nu} \left[ \left( \sum_{\sigma} G_{\sigma\nu} \frac{n_{\sigma} \mathbf{v}_{\sigma}}{c} \right) \times \nabla \times \mathbf{J}_{\nu} \right], \end{aligned} \quad (5.13)$$

where  $\mathbf{J} \equiv (c/4\pi) \nabla \times \mathbf{B} = \sum_{\sigma} q_{\sigma} \mathbf{J}_{\sigma}$  is the current density flowing in a time-independent magnetized plasma. The first term on the right side of (5.13) represents the classical term associated with equilibrium in a magnetized plasma. The second and third terms denote first-order neutrino-plasma contributions to magnetic-field equilibrium.

Shukla *et al.* [30] derived a similar equilibrium condition with only the electrostatic-like term present on the right side (5.13). For a primordial plasma, using (1.7), we note that the neutrino-induced electrostatic-like term once again vanishes from the magnetic-field generation picture. Hence, whereas the second term in (5.13) vanishes for a primordial plasma, the third term on the right side of (5.13), however, does not. Magnetic equilibrium in a primordial neutrino-plasma is thus described by the balance equation

$$\sum_{\sigma} \frac{\mathbf{J}_{\sigma}}{c} \times \left( q_{\sigma} \mathbf{B} + \sum_{\nu} G_{\sigma\nu} \nabla \times \mathbf{J}_{\nu} \right) = \nabla \cdot \left[ \sum_{s=\sigma, \bar{\sigma}} (n_s \mathbf{v}_s \mathbf{P}_s + \mathbf{I} p_s) \right], \quad (5.14)$$

where summation over species on the left side of (5.14) involves only particle species, while the summation on the right side involves particle and anti-particle species. Once again, neutrino-flux vorticity  $\nabla \times \mathbf{J}_{\nu}$  plays a fundamental role in collective neutrino-plasma interactions in the presence of an electromagnetic field.

## VI. SUMMARY AND FUTURE WORK

We now summarize our work and discuss future work. The model for collective neutrino-plasma interactions presented in this work is based on the nonlinear dissipationless fluid equations (4.4), (4.9) and (4.18). These equations are derived from a variational principle based on the relativistic covariant Lagrangian density (3.5). An exact energy-momentum conservation law (4.28) is obtained by Noether method with the energy-momentum tensor for self-consistent collective neutrino-plasma interactions in the presence of an electromagnetic field is given by (4.29). New ponderomotive forces acting on the plasma-neutrino fluids, which are absent from previous works [14,30,31], are given by (4.5) and (4.10) [or (4.7) and (4.12), respectively]. In Eqs. (5.7) and (5.13), we have demonstrated the crucial role played by neutrino-flux vorticity ( $\nabla \times \mathbf{J}_{\nu}$ ) in the processes of magnetic-field generation and magnetic-helicity production in neutrino-plasma fluids.

In future work, we plan to further investigate the importance of the new neutrino-induced ponderomotive terms associated with neutrino fluxes. For this purpose, it might also be useful to derive from ideal neutrino-magnetohydrodynamic equations from (4.4), (4.9) and (4.18). Using the new mechanisms for magnetic-field generation and magnetic-helicity production proposed in (5.7) and (5.12), respectively, we plan to investigate the

problem of magnetogenesis in the early universe. As another application, we plan to investigate neutrino-plasma three-wave interactions leading to the excitation of various plasma waves in unmagnetized and magnetized plasmas; such transfer processes could be important during supernova explosions.

## ACKNOWLEDGMENTS

This work was performed under Department of Energy Contracts No. PDDEFG-03-95ER-40936 and DE-AC03-76SF00098, in part by the National Science Foundation under grant PHY-95-14797, and in part also by Alfred P. Sloan Foundation.

## APPENDIX A: DIFFERENTIAL GEOMETRIC FORMULATION OF CONSTRAINED VARIATIONS

In this Appendix, the geometric interpretation of the constrained variations (3.10, 3.12, 3.14, 3.16) is given in terms of Lie derivatives along the virtual displacement four-vector  $\delta\xi$ . Since the variation of a fluid field is only its infinitesimal displacement, all covariant quantities are varied by their Lie derivatives with respect to the virtual displacement four-vector  $\delta\xi$ . Here, we use the following definition of the Lie derivative on the  $k$ -form  $\alpha$  along the four-vector  $\delta\xi$ , denoted  $\mathbf{L}_{\delta\xi}\alpha$  [51]:

$$\mathbf{L}_{\delta\xi}\alpha \equiv \mathbf{i}_{\delta\xi} \cdot d\alpha + d(\mathbf{i}_{\delta\xi} \cdot \alpha). \quad (\text{A1})$$

Here,  $d\alpha$  is a  $(k+1)$ -form while  $\mathbf{i}_{\delta\xi} \cdot \alpha$  is a  $(k-1)$ -form representing the contraction of the four-vector  $\delta\xi$  with the  $k$ -form  $\alpha$ . By definition, if  $\alpha = \varphi$  is a scalar field (i.e., a zero-form),  $\mathbf{i}_{\delta\xi} \cdot \varphi \equiv 0$ .

The constrained variation  $\delta S = -\delta\xi \cdot \partial S$  for the entropy  $S$  [(3.16)] is consistent with its geometric interpretation as a scalar field:

$$\delta S \equiv -\mathbf{L}_{\delta\xi}S = -\delta\xi \cdot \partial S, \quad (\text{A2})$$

where  $\mathbf{i}_{\delta\xi} \cdot S \equiv 0$  and  $\mathbf{i}_{\delta\xi} \cdot dS \equiv (\delta\xi \cdot \partial)S$ .

The geometric interpretation of the particle flux  $J^\alpha \equiv Nu^\alpha$  is given as the components of the three-form  $J = (1/3!)\epsilon_{\alpha\beta\kappa\lambda}J^\alpha dx^\beta dx^\kappa dx^\lambda$ . The constrained variation of the particle-flux four-vector is defined as

$$\delta J \equiv -\mathbf{L}_{\delta\xi}J. \quad (\text{A3})$$

Since  $dJ \equiv (\partial \cdot J)\mathbf{\Omega}$  with the volume four-form  $\mathbf{\Omega} \equiv dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ , and hence  $dJ = 0$  due to the continuity equation, we obtain  $\delta J = -d(\mathbf{i}_{\delta\xi} \cdot J)$ , or

$$\delta J^\alpha = \partial_\beta (J^\beta \delta\xi^\alpha - J^\alpha \delta\xi^\beta), \quad (\text{A4})$$

which is Eq. (3.14) itself. From this variation, one can easily compute the variations of  $N = \sqrt{J^\alpha J_\alpha}$  and  $u^\alpha = J^\alpha/N$  leading (3.12) and (3.10), respectively.

In Sec. IV D, we consider infinitesimal translations  $x^\alpha \rightarrow x^\alpha + \delta x^\alpha$  generated by the infinitesimal displacement four-vector  $\delta x$ . Under this transformation, the Lagrangian density  $\mathcal{L}$  changes by  $\delta\mathcal{L} \equiv -\partial \cdot (\delta x \mathcal{L})$ . This expression is consistent with the geometric interpretation of  $\mathcal{L}$  as a density in four-dimensional space, i.e.,

$$\delta\mathcal{L} \Omega \equiv -\mathbf{L}_{\delta x}(\mathcal{L} \Omega), \quad (\text{A5})$$

where  $\mathbf{L}_{\delta x}$  is the Lie derivative with respect to  $\delta x$ . Here, using  $\mathbf{i}_{\delta x} \cdot d(\mathcal{L} \Omega) = 0$  and

$$d[\mathbf{i}_{\delta x} \cdot (\mathcal{L} \Omega)] = d(\mathcal{L} \delta x \cdot \omega) \equiv \partial \cdot (\delta x \mathcal{L}) \Omega, \quad (\text{A6})$$

we easily recover (4.21).

Next, the expressions for  $\delta A$  is given in (4.22). Here, the electromagnetic four-potential  $A$  appears as the components of the one-form  $A \cdot dx$ . Thus

$$\delta A \cdot dx \equiv -\mathbf{L}_{\delta x}(A \cdot dx). \quad (\text{A7})$$

Since  $\mathbf{i}_{\delta x} \cdot d(A \cdot dx) = -(\mathbf{F} \cdot \delta x) \cdot dx$  and  $d[\mathbf{i}_{\delta x} \cdot (A \cdot dx)] = d(A \cdot dx)$ , we easily recover (4.22) for the four-potential  $A$ . We note that the expression  $\delta\xi \equiv \mathbf{h} \cdot \delta x$  given in (4.22) is consistent with the expressions  $\delta S = -\mathbf{L}_{\delta\xi} S \equiv -\mathbf{L}_{\delta x} S$  and  $\delta J \cdot \omega = -\mathbf{L}_{\delta\xi}(J \cdot \omega) \equiv -\mathbf{L}_{\delta x}(J \cdot \omega)$ .

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